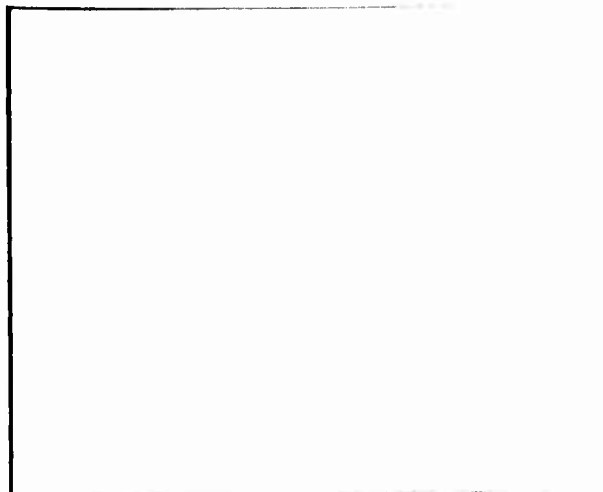


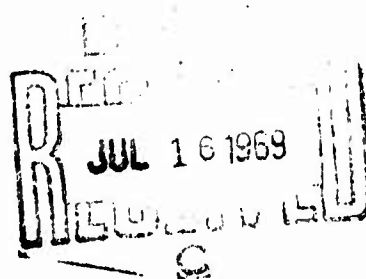
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TULLOCK ON VOTING AND OTHER ASPECTS OF  
POLITICS

BY

Kenneth J. Arrow

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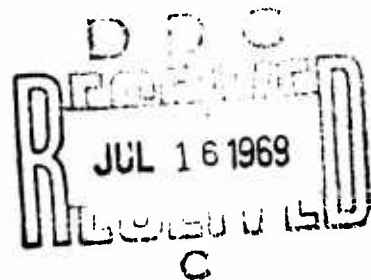
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TULLOCK ON VOTING AND OTHER ASPECTS OF  
POLITICS

Kenneth J. Arrow

Gordon Tullock's latest book<sup>1</sup> is a continuation of the now flourishing tradition which seeks to explain the political process in terms of the rational behavior of its participants, the voters and the politicians. What is sought is a theory analogous to that which has dominated economics, in which the observed behavior of all is found as an equilibrium in which each participant is maximizing some suitably defined criterion given institutional and technological constraints and the behavior of others. This tradition has had several sources: (1) an intermittent interest in evaluating alternative voting systems, the natural criteria being measures of resemblance between individual preference scales and the social outcome of the voting process; the key names are Borda and Condorcet in the 18th century, Nanson and Dodgson (Lewis Carroll) in the 19th; (2) the interest of the marginal utility theorists in demonstrating that their tools were applicable to wider areas of human behavior than the purely economic (Wicksteed, Marshall, and, in modern times, Becker, serve as examples); (3) closely related, the economic theory of bargain-

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<sup>1</sup>G. Tullock, Towards a Mathematics of Politics. Ann Arbor: University of Michigan Press, 1967, vi + 176 pp. \$7.95.

ing, especially in the form developed by Edgeworth, clearly invites extension to the political and other social spheres, a step taken by Hotelling and Zeuthen and, in very great generality, by von Neumann and Morgenstern and many of the subsequent game theorists (Nash, Shapley, Shubik, Aumann, Harsanyi, and Maschler, among others); and (4) the interest of public finance theorists in finding a public demand analogous to private demand (Wicksell, Lindahl, Bowen, and more recently, Musgrave and Samuelson). Duncan Black's work of the 1940's synthesized several of these intellectual streams and began the continuous tradition which has been further developed by many writers, among whom Downs, Riker, Buchanan, Olson, and Tullock are especially to be mentioned.

The present book is, as the author notes in his preface, not the development of a single theme, but a series of essays on distinct aspects of a rational theory of politics. The title is misleading; there are no developments of any mathematical complexity. The mode of analysis is that of literary economic theory, the logical development being conveyed in words rather than symbols. Though empirical references are frequent, they are casual in nature, not systematic attempts at refutation of precisely-stated hypotheses.

After an introductory chapter which discusses the nature of preferences and changes in them, the book falls into four main parts. In Chapters II-IV, the implications of majority

voting for the behavior of political parties (assumed to maximize votes) are explored. In Chapters V and VI, there is some study of the implications of single-peaked preferences for monopolistic competition and, in particular, for the production of information. Chapters VII-IX analyze the production and consumption of information in the political sphere, with emphasis on persuasion by true or false information. Finally, Chapter X is an essay on proportional representation, the chief novelty of which is an ingenious proposal by the author that every elected representative cast as many votes in the legislature as he received in the election.

Chapter III presents what I feel is the major analytic contribution of the book, an argument that, under certain plausible hypotheses, the Condorcet voting paradox (the intransitivity of majority voting) will not arise. In view of the importance of this result, I want to present a general formulation of Tullock's proposition and some comments on its applicability. My comments on the remainder of the book will therefore be brief.

The basic model throughout the book is that of a multi-dimensional space of social issues over which each individual has a preference ordering. In Chapter II, a geometric analysis of the two-dimensional case is carried through with great care. This chapter is a superb piece of pedagogy. As a result of careful examination of a number of examples, it is concluded that when the number of choosers greatly exceeds the dimensionality of

of the issue space, then the set of Pareto-optimal points is apt to be a fairly large proportion of the total space; the interesting question then is the choice within the Pareto region, and a voting rule which tends to pick out some average point is desirable. If, on the other hand, the dimensionality of the issue space is large compared with the number of choosers, the Pareto region is relatively small; simply attaining a point in it becomes important, so that the rule of unanimity is powerful. The former case, Tullock identifies with the domain of politics, the latter with that of economics (the market).

Chapter IV attempts to extend the Hotelling-Downs model of competition among parties to two-dimensional issue spaces; a number of examples are given, suggesting that the tendency to a median is frequently still valid.

Chapter V, on monopolistic competition, is mainly devoted to an argument that the monopolistically competitive equilibrium may be close to optimal. The author here suffers from a lack of reference to the literature; even Chamberlain's later editions made this point, and in any case no rigorous analysis or even formulation of notion of a monopolistically competitive equilibrium has yet been made. Chapter VI has a number of interesting comments on the incentive for the provision of different kinds of information (here, principally editorial opinion) in response to demand, but no strong generalizations appear.

Chapters VII-IX expand on a theme introduced into the liter-

ature by Anthony Downs; since the effect of any individual vote is so very small, it does not pay a voter to acquire information unless his stake in the issue is enormously greater than the cost of information. The resulting rational ignorance in the political field in turn creates incentives to political parties and other strongly motivated groups to engage in persuasive activities by dissemination of information, selected or even false. Tullock's discussion is discursive; a number of interesting points are made which do not lend themselves to easy summary.

Let me return to Chapter III. To quote, "A phantom has stalked the classrooms and seminars of economics and political science for nearly fifteen years. This phantom, Arrow's General Impossibility Theorem, has been generally interpreted as proving that no sensible method of aggregating preferences exists. The purpose of this essay is to exorcise the phantom, not by disproving the theorem in its strict mathematical form, but by showing that it is insubstantial. I shall show that when a rather simple and probable type of interdependence is assumed among the preference functions of the choosing individuals, the problem becomes trivial if the number of voters is large. Since most cases which require aggregation of preferences involve large numbers of people, 'Arrow problems' will seldom be of much importance."

What Tullock actually does, in effect, is to give an example, considered to be typical of real-world situations, where there is one point in the social issue space which is preferred by a majority

to all others. He assumes (a) that the number of voters is large, so large that we may consider them to constitute a continuum, (b) that the indifference curves of each individual in the space of social issues are circles concentric about a global optimum (as seen by the individual), and (c) the global optima of the different voters are uniformly distributed over the issue space, taken to be a rectangle. From (b), an individual will prefer point A to point B if and only if his global optimum is closer to A than to B. If we draw the perpendicular of the line joining the two points and thus divide the rectangle into two regions, then those individuals whose global optima are in the region containing A are precisely those who prefer A to B. In view of (c), then, a majority prefer A to B if the area of the region containing A exceeds that of the region containing B.

It is then easy to infer that A is preferred to B if and only if it is closer to the center of the rectangle. Since the relation, "closer to the center," is certainly transitive, it follows that majority decision yields a true ordering under these circumstances. In particular, the center of the rectangle is preferred to any other alternative.

The question needs to be raised, does this example generalize in any meaningful way? It so happens that a generalization can be found by making use of some results of an unpublished paper by Sonnenschein.<sup>2</sup>

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<sup>2</sup>H. Sonnenschein, "Demand Functions without Transitive Preferences, with Applications to the Theory of Competitive Equilibrium," unpublished manuscript, University of Minnesota.



Sonnenschein's interest was in a seemingly remote problem; was it possible to get the usual results in the theory of consumer's demand without assuming the transitivity of the preference relation. His answer was, in effect, that it was provided another hypothesis was made, that the set of alternatives preferred to a given alternative is always convex.

Sonnenschein's result immediately suggests application to (weak) majority choice, a relation which is connected but not necessarily transitive. For any pair of alternatives  $x, y$ , let  $N(x, y)$  be the number of individuals who prefer  $x$  to  $y$ . Then let  $x M y$  be the statement,  $N(x, y) \geq N(y, x)$ ,  $x \bar{M} y$  the statement that  $N(x, y) > N(y, x)$ . The question then is, given a set of alternatives, say  $S$ , under what conditions does there exist an alternative chosen by majority decision, i.e., an alternative  $x$  in  $S$  such that  $x M y$  for all  $y$  in  $S$ ? An answer is provided by the following

Theorem. Suppose that, for each alternative  $x^0$ , the set of alternatives  $x$  for which  $x M x^0$  is closed and the set of alternatives for which  $x \bar{M} x^0$  is convex. Then for any compact (i.e., closed and bounded) convex set of alternatives,  $S$ , there is (at least) one alternative  $x$  in  $S$  such that  $x M y$  for all  $y$  in  $S$ .

Although the proof is simply a transcription of that in Sonnenschein's paper, it may be worth extracting the essential points from the consumer's demand context. The proof is based on the following two mathematical lemmas valid for finite dimensional

Euclidean spaces.

Lemma 1. Let  $S$  be a compact set and  $F$  a family of closed subsets of  $S$  such that every finite collection of sets in  $F$  has an element in common. Then there is an element common to all the sets in  $F$ .

Lemma 2. For each  $i = 1, \dots, m$ , let  $a_i$  be a point and  $S_i$  a closed set. Suppose that for any subset  $L$  of the indices  $1, \dots, m$ , any convex combination of the  $a_i$ 's ( $i \in L$ ) belongs to at least one  $S_i$  with  $i \in L$ . Then there is a convex combination of the  $a_i$ 's ( $i=1, \dots, m$ ) which belongs to all of the  $S_i$ 's ( $i=1, \dots, m$ ).

Lemma 1 is a standard proposition in real analysis. Lemma 2 is a slight generalization of the well-known Knaster-Kuratowski-Mazurkiewicz lemma in topology, a proposition closely related to Brauer's fixed point theorem.

Proof of Theorem. Consider any given compact convex set of alternatives,  $S$ ; for any  $y$  in  $S$ , let  $S(y)$  be the set of alternatives  $x$  in  $S$  for which  $x M y$ . By hypothesis, the set of all alternatives  $x$  for which  $x M y$  is a closed set, and therefore the set of alternatives for which both  $x$  in  $S$  and  $x M y$  is also a closed set, since  $S$  is closed.

Let  $a_i$  ( $i=1, \dots, m$ ) be any  $m$  elements of  $S$ , and let  $L$  be any subset of the indices  $1, \dots, m$ . Let  $y$  be a convex combination of the points  $a_i$  ( $i \in L$ ); it is first shown that  $y \in S(a_i)$  for some  $i \in L$ . For suppose not; then, from the definition, it must be that  $y M a_i$

is false for all  $i$  in  $L$ , i.e.,  $a_i \bar{M} y$  for all  $i$  in  $L$ . But  $y$  is a convex combination of the points  $a_i$ ; by hypothesis, the set of alternatives  $x \bar{M} y$  is convex, so that  $y \bar{M} y$ , an obvious contradiction. Hence, any convex combination of the points  $a_i$  ( $i \in L$ ) belongs to at least one  $S(a_i)$  with  $i \in L$ . From Lemma 2, it then follows immediately that there is at least one point common to all the sets  $S(a_i)$ ,  $i = 1, \dots, m$ .

Let  $F$  be the family of all sets  $S(y)$  as  $y$  varies over  $S$ . It has just been shown that any finite collection of sets in  $F$  has an element in common. By Lemma 1, there is a point,  $x$ , which is common to all the sets  $S(y)$ . That is,  $x M y$  for all  $y$  in  $S$ ; since  $x \in S(y)$ , it certainly belongs to  $S$ , by definition. Q.E.D.

The hypotheses of the theorem are obviously fulfilled in Tullock's example.

Several remarks are in order to help clarify the meaning of this theorem.

Remark 1. The conclusion of the theorem is something weaker than transitivity; it asserts the existence of a preferred point (more strictly a non-inferior point) in any convex set. This is an important restriction, since all cases of increasing return are ruled out of consideration, and these are, after all, a major portion of the scope of public decision-making. It is not, however, absolutely clear that this restriction is in fact necessary. In Tullock's example, transitivity held, so that there would exist a non-inferior point (according to the majority decision criterion)

from any compact set of alternatives. I would conjecture that the conclusion of the theorem cannot be improved on without further hypotheses, but I have no example to prove this .

Remark 2. The hypothesis of convexity of the majority preference sets is, in some rough sense, a hypothesis of similarity of attitudes, but its exact meaning is obscure. It would be useful to find sufficient conditions for the convexity of majority preference sets, conditions which might be stronger but which are more transparent. One condition can be given which is transparent but perhaps not yet sufficiently so: for any fixed  $x_0$ , the function  $N(x, x_0) - N(x_0, x)$  is quasi-concave. First, note that this condition means that the set of  $x$ 's for which  $N(x, x_0) - N(x_0, x) > c$  is convex for any  $c$  and, in particular, for  $c = 0$ , which defines the majority preference set. To understand the meaning of this condition, take any straight line in the space of alternatives. For any  $x$  on the line, we can ask each voter if he prefers  $x$  to the given alternative,  $x_0$ ; the statement that the function,  $N(x, x_0) - N(x_0, x)$ , is quasi-concave amounts to saying that the number of voters who answer, "yes," to the above question increases as we move along the line up to a maximum and then decreases (or the number may increase all the way or decrease all the way). This does appear to me to convey an idea of similarity of preferences without imposing absolute identity.

Remark 3. Perhaps the deepest impulse to the study of the theory of social choice at least to the economist, is the hope of

saying something useful about the evolution of income distributions. It should be made clear that the Theorem of this review note does not directly help at all in the resolution of this problem. Suppose a number of individuals with completely egoistic preferences use the method of majority decision to divide up a fixed total of a single commodity. Then, as has been remarked many times, there is no allocation which will command a majority against any other. For the allocation which gives zero to all is inferior, by unanimous vote, to one which gives a positive amount to each chosen sufficiently small to be feasible. For any allocation which gives some individual, say 1, a positive amount, there is another, which gives 1 nothing and divides up his share in the first allocation among all the others; the second is preferred to the first by all but one individual. It can easily be shown in this case that the majority preference sets are not convex.

In this case of unrestricted income distribution, the dimensionality of the issue space is the same as the number of individuals. Thus, as Tullock argues, political resolution of distributional issues is apt to be possible only if only a few parameters of the income distribution are under consideration, not the whole distribution. Why this restriction of the scope of choice should occur is not easy to explain on simple economic grounds. On the other hand, the restriction does conform to the long-standing view of writers on ethics, of who Kant is perhaps most conspicuous, that decisions on distribution ought to be made as if by an impartial

observer, who considers then only the mean, a measure of inequality, and perhaps one or two further parameters characterizing the income distribution, but not specifically who gets what. If voters acted like Kantian judges, they might still differ, but the chances of coming to an agreement by majority decision would be much greater than if voters consulted egoistic values only. Does this suggest that ethics may have survival value for political systems and therefore descriptive as well as prescriptive significance?

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